```
2 - 2
```

Out[268]= 0

Sebbar's Points of Departure

Nicholas Wheeler
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Introduction

Yesterday, Ahmed Sebbar sent me a note in which he sketched the considerations that led to his interest in what I call "Sebbar polynomials," but which he suggests might be called "Pincherle polynomials" because some special cases were discussed in a 1891 paper by Salvatore Pincherle. Here I reproduce Sebbar's hurried remarks.

Motivational considerations

Introduce the Laplacian operator

In[270]:=
```
Δ[f_] := D[f, {x, 2}] + D[f, {y, 2}]
```

and observe that $\text{Log}\left[x^2 + y^2\right]$ is a solution of Laplace's equation:

```
Δ[Log[x^2 + y^2]] // Simplify
```

Out[275]= 0

Shift along the x-axis:

In[279]:=
```
Δ[Log[(x - h)^2 + y^2]] // Simplify
```

Out[279]= 0

Evaluate

In[280]:=
```
(x - h)^2 + y^2 // Expand
```

Out[280]= $h^2 - 2\,h\,x + x^2 + y^2$

on the unit circle:

In[281]:=
```
(x - h)^2 + y^2 /. y^2 → 1 - x^2 // Expand
```

Out[281]= $1 + h^2 - 2\,h\,x$

So it is shifted constraint to the unit circle that produces the $1 + h^2 - 2\,h\,x$ that figures in two of Sebbar's polynomials.

Look now (why?) to the 3-variable construction

In[337]:=
```
f = x^3 + y^3 + z^3 - 3 x y z;
```

Again shift along the x-axis

In[338]:=
```
f /. x → x - h // Expand
```

Out[338]= $-h^3 + 3\,h^2\,x - 3\,h\,x^2 + x^3 + y^3 + 3\,h\,y\,z - 3\,x\,y\,z + z^3$

and constrain (why?…beyond the fact that it does simplify things) to the curve produced by intersection of the surfaces

```
x^3 + y^3 + z^3 - 3 x y z = 1

y z - x^2 = 0
```

which are respectively a unit hexenhut and a cone. We have

```
In[351]:= -h^3 + 3 h^2 x - 3 h x^2 + x^3 + y^3 + 3 h y z - 3 x y z + z^3 ==
            -h^3 + 3 h^2 x - 3 h x^2 + x^3 + (y^3 + z^3) + 3 (h - x) y z // Simplify

Out[351]= True
```

which by the constraint relations becomes

```
In[395]:= -h^3 + 3 h^2 x - 3 h x^2 + x^3 + (1 - x^3 + 3 x^3) + 3 (h - x) x^2 // Simplify

Out[395]= 1 - h^3 + 3 h^2 x
```

The following adjustment

```
In[396]:= 1/h^3 % // Simplify

Out[396]= -1 + 1/h^3 + (3 x)/h

In[397]:= -% /. {1/h^3 → g^3, 1/h → g}

Out[397]= 1 - g^3 - 3 g x
```

changes the coefficient of x from $3\, h^2$ to $-3\, g^1$. Compare this with the result to which the 2-dimensional theory led:

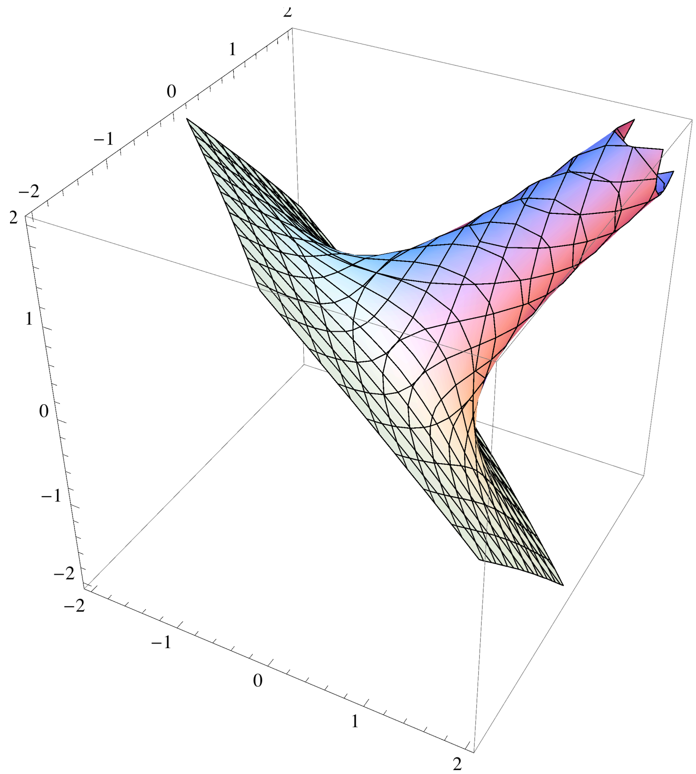
$1 + h^2 - 2\, h\, x$

■ Geometry of the situation

The first constraint produces the hexenhut

In[299]:=
```
ContourPlot3D[{x^3 + y^3 + z^3 - 3 x y z == 1}, {x, -2, 2}, {y, -2, 2}, {z, -2, 2}]
```

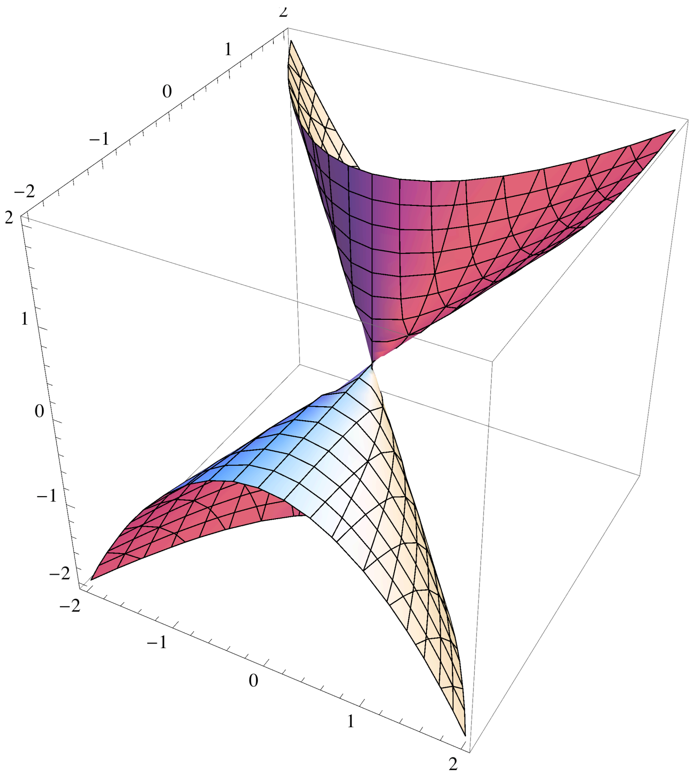
Out[299]=



and the second constraint produces a cone

In[300]:=
```
ContourPlot3D[{y z - x^2 == 0}, {x, -2, 2}, {y, -2, 2}, {z, -2, 2}]
```

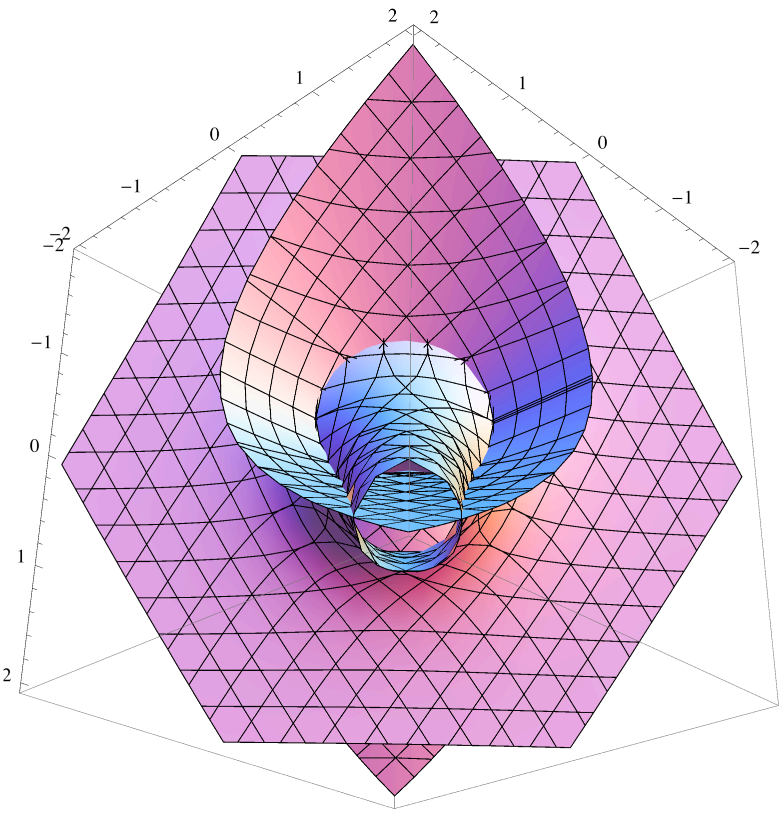
Out[300]=



The following figure shows their intersection:

In[326]:=
```
SebbarSurfaces =
  ContourPlot3D[{x^3 + y^3 + z^3 - 3 x y z == 1, y z - x^2 == 0}, {x, -2, 2}, {y, -2, 2}, {z, -2, 2}]
```

Out[326]=



The solution of the intersection equations has six branches

In[306]:= **Solve**$\left[\left\{x^3+y^3+z^3-3\,x\,y\,z==1,\ y\,z-x^2==0\right\},\ \{y,z\}\right]$

Out[306]=
$$\left\{\left\{y\to\frac{1}{x^4}\left(\frac{\left(1+2\,x^3-\sqrt{1+4\,x^3}\right)^{2/3}}{2\,2^{2/3}}+\frac{x^3\left(1+2\,x^3-\sqrt{1+4\,x^3}\right)^{2/3}}{2^{2/3}}+\frac{\sqrt{1+4\,x^3}\left(1+2\,x^3-\sqrt{1+4\,x^3}\right)^{2/3}}{2\,2^{2/3}}\right),\right.\right.$$
$$\left.z\to\frac{\left(1+2\,x^3-\sqrt{1+4\,x^3}\right)^{1/3}}{2^{1/3}}\right\},$$
$$\left\{y\to\frac{1}{x^4}\left(-\frac{(-1)^{1/3}\left(1+2\,x^3-\sqrt{1+4\,x^3}\right)^{2/3}}{2\,2^{2/3}}-\frac{(-1)^{1/3}\,x^3\left(1+2\,x^3-\sqrt{1+4\,x^3}\right)^{2/3}}{2^{2/3}}-\right.\right.$$
$$\left.\left.\frac{(-1)^{1/3}\sqrt{1+4\,x^3}\left(1+2\,x^3-\sqrt{1+4\,x^3}\right)^{2/3}}{2\,2^{2/3}}\right),\ z\to\frac{(-1)^{2/3}\left(1+2\,x^3-\sqrt{1+4\,x^3}\right)^{1/3}}{2^{1/3}}\right\},$$
$$\left\{y\to\frac{1}{x^4}\left(\frac{1}{2}\left(-\frac{1}{2}\right)^{2/3}\left(1+2\,x^3-\sqrt{1+4\,x^3}\right)^{2/3}+\left(-\frac{1}{2}\right)^{2/3}x^3\left(1+2\,x^3-\sqrt{1+4\,x^3}\right)^{2/3}+\right.\right.$$
$$\left.\left.\frac{(-1)^{2/3}\sqrt{1+4\,x^3}\left(1+2\,x^3-\sqrt{1+4\,x^3}\right)^{2/3}}{2\,2^{2/3}}\right),\ z\to-\left(-\frac{1}{2}\right)^{1/3}\left(1+2\,x^3-\sqrt{1+4\,x^3}\right)^{1/3}\right\},$$
$$\left\{y\to\frac{1}{x^4}\left(\frac{1}{2}\left(\frac{1}{2}+x^3+\frac{1}{2}\sqrt{1+4\,x^3}\right)^{2/3}+x^3\left(\frac{1}{2}+x^3+\frac{1}{2}\sqrt{1+4\,x^3}\right)^{2/3}-\right.\right.$$
$$\left.\left.\frac{1}{2}\sqrt{1+4\,x^3}\left(\frac{1}{2}+x^3+\frac{1}{2}\sqrt{1+4\,x^3}\right)^{2/3}\right),\ z\to\left(\frac{1}{2}+x^3+\frac{1}{2}\sqrt{1+4\,x^3}\right)^{1/3}\right\},$$
$$\left\{y\to\frac{1}{x^4}\left(-\frac{1}{2}\,(-1)^{1/3}\left(\frac{1}{2}+x^3+\frac{1}{2}\sqrt{1+4\,x^3}\right)^{2/3}-(-1)^{1/3}\,x^3\left(\frac{1}{2}+x^3+\frac{1}{2}\sqrt{1+4\,x^3}\right)^{2/3}+\right.\right.$$
$$\left.\left.\frac{1}{2}\,(-1)^{1/3}\sqrt{1+4\,x^3}\left(\frac{1}{2}+x^3+\frac{1}{2}\sqrt{1+4\,x^3}\right)^{2/3}\right),\ z\to(-1)^{2/3}\left(\frac{1}{2}+x^3+\frac{1}{2}\sqrt{1+4\,x^3}\right)^{1/3}\right\},$$
$$\left\{y\to\frac{1}{x^4}\left(\frac{1}{2}\,(-1)^{2/3}\left(\frac{1}{2}+x^3+\frac{1}{2}\sqrt{1+4\,x^3}\right)^{2/3}+(-1)^{2/3}\,x^3\left(\frac{1}{2}+x^3+\frac{1}{2}\sqrt{1+4\,x^3}\right)^{2/3}-\right.\right.$$
$$\left.\left.\left.\frac{1}{2}\,(-1)^{2/3}\sqrt{1+4\,x^3}\left(\frac{1}{2}+x^3+\frac{1}{2}\sqrt{1+4\,x^3}\right)^{2/3}\right),\ z\to-(-1)^{1/3}\left(\frac{1}{2}+x^3+\frac{1}{2}\sqrt{1+4\,x^3}\right)^{1/3}\right\}\right\}$$

of which ony two are real, and it is only in those that we have interest:

In[353]:= **Branch1 =**

$$\left\{y \to \frac{1}{x^4}\left(\frac{\left(1+2x^3-\sqrt{1+4x^3}\right)^{2/3}}{2\,2^{2/3}} + \frac{x^3\left(1+2x^3-\sqrt{1+4x^3}\right)^{2/3}}{2^{2/3}} + \frac{\sqrt{1+4x^3}\left(1+2x^3-\sqrt{1+4x^3}\right)^{2/3}}{2\,2^{2/3}}\right),\ z \to \frac{\left(1+2x^3-\sqrt{1+4x^3}\right)^{1/3}}{2^{1/3}}\right\};$$

$$\text{Branch2} = \left\{y \to \frac{1}{x^4}\left(\frac{1}{2}\left(\frac{1}{2}+x^3+\frac{1}{2}\sqrt{1+4x^3}\right)^{2/3} + x^3\left(\frac{1}{2}+x^3+\frac{1}{2}\sqrt{1+4x^3}\right)^{2/3} - \frac{1}{2}\sqrt{1+4x^3}\left(\frac{1}{2}+x^3+\frac{1}{2}\sqrt{1+4x^3}\right)^{2/3}\right),\ z \to \left(\frac{1}{2}+x^3+\frac{1}{2}\sqrt{1+4x^3}\right)^{1/3}\right\};$$

From

In[356]:= $\sqrt{1+4x^3}$ /. $x \to -\frac{1}{2^{2/3}}$

Out[356]= 0

we see that reality of those branches requires $x \geqslant -\frac{1}{2^{2/3}}$. We arrive thus at the parametric description of two curves (branches/halves of the surface-intersection curve):

In[359]:= $$A = \left\{x,\ \frac{1}{x^4}\left(\frac{\left(1+2x^3-\sqrt{1+4x^3}\right)^{2/3}}{2\,2^{2/3}} + \frac{x^3\left(1+2x^3-\sqrt{1+4x^3}\right)^{2/3}}{2^{2/3}} + \frac{\sqrt{1+4x^3}\left(1+2x^3-\sqrt{1+4x^3}\right)^{2/3}}{2\,2^{2/3}}\right),\ \frac{\left(1+2x^3-\sqrt{1+4x^3}\right)^{1/3}}{2^{1/3}}\right\};$$

$$B = \left\{x,\ \frac{1}{x^4}\left(\frac{1}{2}\left(\frac{1}{2}+x^3+\frac{1}{2}\sqrt{1+4x^3}\right)^{2/3} + x^3\left(\frac{1}{2}+x^3+\frac{1}{2}\sqrt{1+4x^3}\right)^{2/3} - \frac{1}{2}\sqrt{1+4x^3}\left(\frac{1}{2}+x^3+\frac{1}{2}\sqrt{1+4x^3}\right)^{2/3}\right),\ \left(\frac{1}{2}+x^3+\frac{1}{2}\sqrt{1+4x^3}\right)^{1/3}\right\};$$

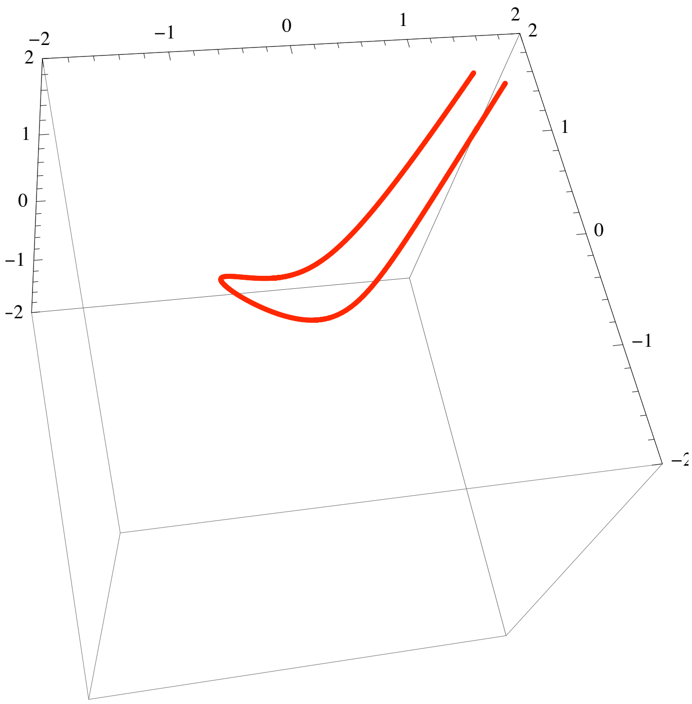
When plotting those curves we must arrange to omit the y-singularity at x = 0. We proceed

In[387]:=
```
𝒜1 = ParametricPlot3D[A, {x, -1/2^(2/3), -0.005},
   PlotRange → {{-2, 2}, {-2, 2}, {-2, 2}}, PlotStyle → {Red, Thickness[0.008]}];
𝒜2 = ParametricPlot3D[A, {x, 0.005, 2}, PlotRange → {{-2, 2}, {-2, 2}, {-2, 2}},
   PlotStyle → {Red, Thickness[0.008]}];
ℬ1 = ParametricPlot3D[B, {x, -1/2^(2/3), -0.005}, PlotRange → {{-2, 2}, {-2, 2}, {-2, 2}},
   PlotStyle → {Red, Thickness[0.008]}]; ℬ2 = ParametricPlot3D[B, {x, 0.005, 2},
   PlotRange → {{-2, 2}, {-2, 2}, {-2, 2}}, PlotStyle → {Red, Thickness[0.008]}];
```

which when joined together produce the following bent hairpin:

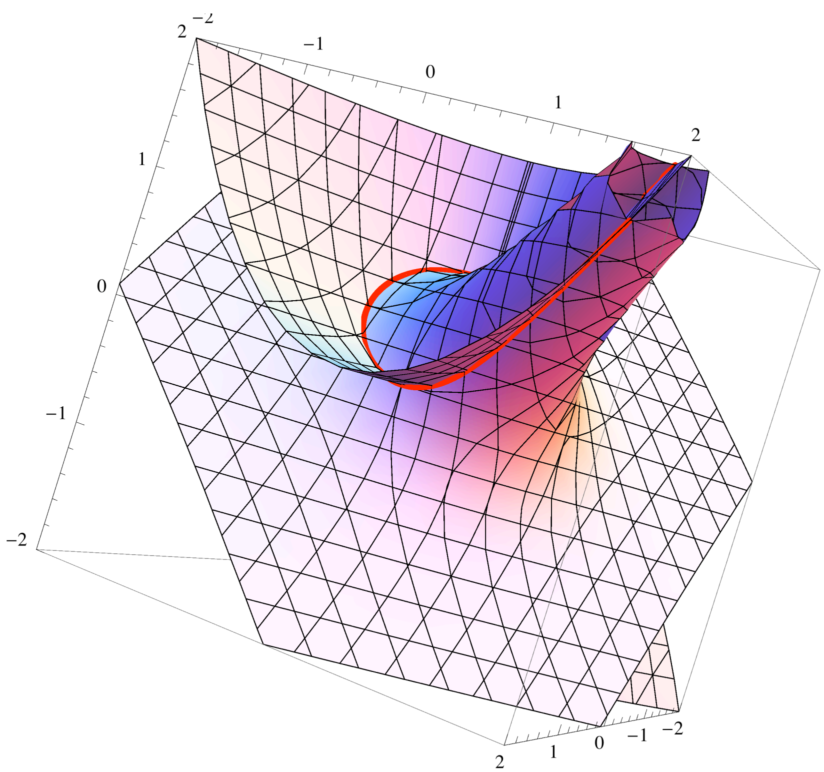
In[389]:= **`Show[𝒜1, 𝒜2, ℬ1, ℬ2]`**

Out[389]=



In[391]:= **`Show[𝒜1, 𝒜2, ℬ1, ℬ2, SebbarSurfaces]`**

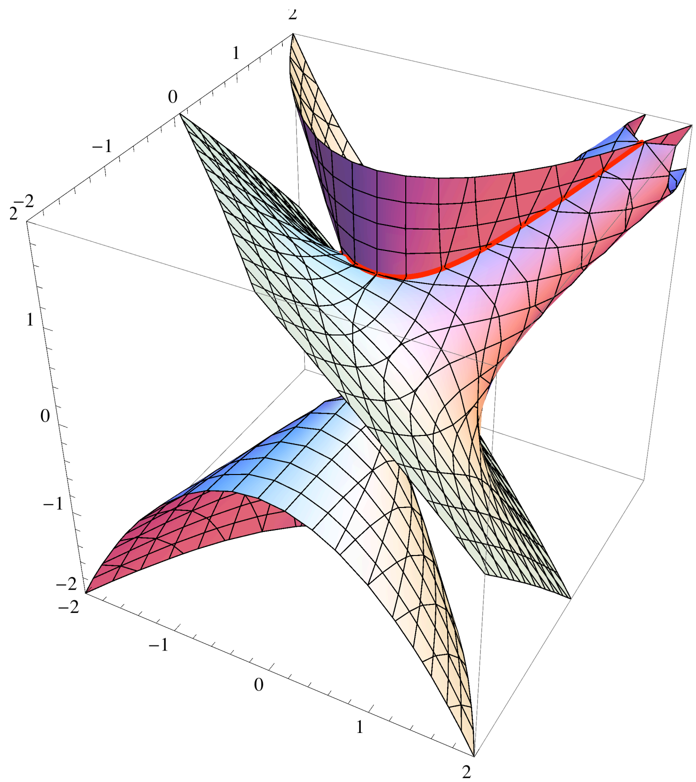
Out[391]=



Another view of the same figure:

In[392]:=
```
Show[𝒜1, 𝒜2, ℬ1, ℬ2, SebbarSurfaces]
```

Out[392]=



- **NOTE:**

This site

https://en.wikipedia.org/wiki/Salvatore_Pincherle

reports that a selection of 62 of Pincherle's papeers was published in 1954 to honor his Centennial. I see that Pincherle (1854 - 1936) studied & collaborated with some of my favorite people (Betti, Dini, Volterra) and knew a lot about the hypergeometric function and its relatives.